

KNOTS WHICH ARE NOT CONCORDANT TO THEIR REVERSES

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If K is an oriented knot in S^3 , the reverse of K , K^* , is the knot K with its orientation reversed. (This has traditionally been called the inverse of K . We call it the reverse to distinguish it from the inverse to K in the knot concordance group, denoted by $-K$ and represented by the mirror image of K with orientation reversed.) Fox [3] asked for an example of a knot which is not isotopic to its reverse. Trotter provided the first examples in [8]. In this paper we will give examples of knots which are not concordant to their reverses.

Finding such an example is difficult on two accounts. A knot and its reverse have equivalent images in the algebraic concordance group defined by Levine [6]. This follows from the fact that if V is a Seifert matrix for K , then V^t is a Seifert matrix for K^* . In [7] Levine defined a complete set of invariants for the algebraic concordance group, none of which are changed by taking the transpose of an element. A second difficulty is that the branched covers of S^3 branched along K and K^* are identical. Hence a direct application of the techniques of Casson and Gordon [1, 2] will not work.

Our approach is to use the refinement of the Casson–Gordon technique which was developed by Gilmer [4]. Throughout this work we will use the results and notation of [4]. Thanks are due to Pat Gilmer for many informative and helpful conversations, and to Cameron Gordon for pointing out this problem.

1. Statement of results

Consider the knot J illustrated in Figure 1. It is constructed as the boundary of a surface F built by adding two untwisted bands to a disk as indicated. One of the bands is tied in a knot L_1 , the other in a knot L_2 . The main result of this paper, the proof of which is contained in sections 2 and 3, is the following.

THEOREM 1. *If J is concordant to J^* then either $\sigma_{(1/3)}(L_1) = 0$, $\sigma_{(1/3)}(L_2) = 0$, or $\sigma_{(1/3)}(L_1) = \sigma_{(1/3)}(L_2)$.*

Remark. Notice that if either L_1 or L_2 is a slice knot then J is also a slice knot and is certainly concordant to J^* . If $L_1 = L_2$ then J is isotopic

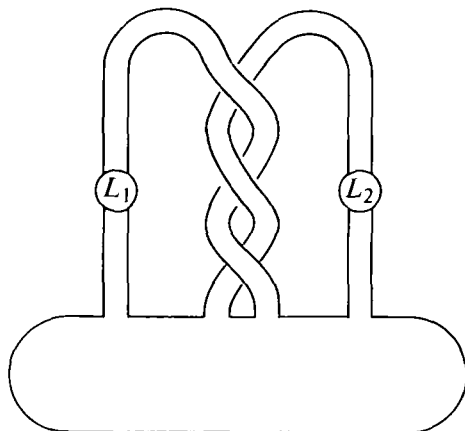


FIG. 1.

to J^* . The isotopy consists of a 180 degree rotation about a vertical axis in the plane of the picture, followed by appropriate twists in each band.

The proof of this theorem consists of showing that if the knot $J\#-J^*$ is slice then one of these three conditions is satisfied. Figure 2 illustrates $J\#-J^*$ along with a basis $\{a, b, c, d\}$ for $H_1(F)$, where F is the evident Seifert surface.

We will now summarize the main result of [4]. Let K be a knot with Seifert surface F . If V is a Seifert matrix for F then V determines a pairing $\theta: H_1(F) \times H_1(F) \rightarrow \mathbb{Z}$ and a symmetric bilinear form on $H_1(F)$, β , given by $V + V^t$. Define $\varepsilon: H_1(F) \rightarrow H^1(F)$ by $\varepsilon(x)(y) = \beta(x, y)$ and let $A = \text{kernel}(\varepsilon \otimes \text{id}(Q/\mathbb{Z}))$. $H^1(L; Q/\mathbb{Z})$ is isomorphic to A , where L is the 2-fold branched cover of S^3 branched over K . Let A' be the subset of

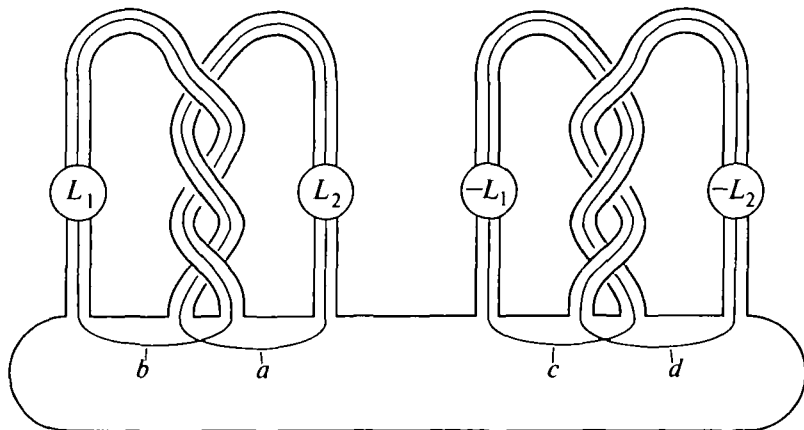


FIG. 2.

elements of A with prime power order, and $\tau(K, \chi)$ the invariant defined by Casson and Gordon [1, 5]. We have the following.

PROPOSITION (Gilmer [4]). *If K is a slice knot there is a direct summand H of $H_1(F)$ such that 1) $2 \operatorname{rank}(H) = \operatorname{rank}(H_1(F))$, 2) $\theta(H \times H) = 0$, and 3) for all $\chi \in A' \cap H \otimes Q/Z$, $\tau(K, \chi) = 0$.*

In section 2 of this paper we will find all summands H satisfying conditions 1 and 2 of the above proposition for the knot $J \# -J^*$. In section 3 the value of $\tau(J \# -J^*, \chi)$ will be computed for the appropriate χ .

2. Null summands for the Seifert form of $J \# -J^*$

With respect to the ordered basis (a, b, c, d) of $H_1(F)$ indicated in Figure 2, the Seifert matrix for $J \# -J^*$ is given by the 4×4 matrix

$$V = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -2 \\ -1 & 0 \end{pmatrix}$$

According to Levine [7] finding 2 dimensional null summands for V is equivalent to finding 2 dimensional subspaces of Q^4 on which the bilinear form β vanishes and which are invariant under the transformation $T = V^{-1}V^t$. Notice that if two direct summands of Z^n have the same span in Q^n they are identical.

In the present case we have

$$T = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \oplus \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

and β is given by the matrix

$$\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$$

The transformation T has eigenvectors $v_1 = (1, 0, 0, 0)$ and $v_2 = (0, 0, 1, 0)$ with eigenvalue 2, and eigenvectors $w_1 = (0, 1, 0, 0)$ and $w_2 = (0, 0, 0, 1)$ with eigenvalue $\frac{1}{2}$. In order to find the invariant subspaces of T in Q^4 we first find the invariant subspaces of T considered as a transformation on C^4 .

LEMMA 1. *Let T be a transformation of a 4 dimensional complex vector space W such that W splits as the direct sum of 2 dimensional eigenspaces with distinct eigenvalues. Any 2 dimensional invariant subspace Y of W is spanned by eigenvectors.*

Proof. As Y is invariant under T , T restricted to Y has an eigenvector,

which must be an eigenvector of T acting on W . Denote it by x_1 . W is spanned by eigenvectors x_1 and x_2 of eigenvalue λ_1 and eigenvectors y_1 and y_2 of eigenvalue λ_2 . Y is hence spanned by x_1 and $a_1x_1 + a_2x_2 + b_1y_1 + b_2y_2$. A change of basis shows that Y is also spanned by x_1 and $a_2x_2 + b_1y_1 + b_2y_2$. As Y is invariant under T , $T(a_2x_2 + b_1y_1 + b_2y_2) = m_1x_1 + m_2(a_2x_2 + b_1y_1 + b_2y_2)$. This gives $\lambda_1a_2x_2 + \lambda_2b_1y_1 + \lambda_2b_2y_2 = m_1x_1 + m_2(a_2x_2 + b_1y_1 + b_2y_2)$. Clearly $m_1 = 0$, so $a_2x_2 + b_1y_1 + b_2y_2$ must be an eigenvector also.

Letting $\langle \rangle$ denote the span of vectors, the possible invariant subspaces of T acting on C^4 are $\langle v_1, v_2 \rangle$, $\langle w_1, w_2 \rangle$, $\langle v_2, \alpha_1w_1 + \alpha_2w_2 \rangle$, $\langle w_2, \beta_1v_1 + \beta_2v_2 \rangle$, and $\langle v_1 + t_1v_2, w_1 + t_2w_2 \rangle$. If we use β to denote the Hermitian form on C^4 determined by $\beta = V + V^t$, we find

$$\beta(v_i, v_j) = 0 \quad \beta(w_i, w_j) = 0 \quad \beta(v_i, w_j) = (-1)^i 3\delta_{i,j}$$

for any i and j , with $\delta_{i,j}$ being 1 or 0 depending on whether or not i equals j .

Applying this calculation immediately gives the following.

LEMMA 2. *The 2 dimensional subspaces of C^4 which are invariant under T and on which β vanishes are: A) $\langle v_1, v_2 \rangle$, B) $\langle w_1, w_2 \rangle$, C) $\langle v_2, w_1 \rangle$, D) $\langle v_1, w_2 \rangle$, and E) $\langle v_1 + tv_2, w_1 + (1/\bar{t})w_2 \rangle$, where t is any nonzero complex number.*

In each of these cases we must now determine all 2 dimensional rational subspaces, and then generators of the associated Z summand of $H_1(F)$. In case A, if $av_1 + bv_2 \in Q^4$ it is clear that both a and b must be rational. The Z direct summand contained in the rational span of v_1 and v_2 is $H_A = \langle (1, 0, 0, 0), (0, 0, 1, 0) \rangle$. Cases B, C, and D are similar, giving $H_B = \langle (0, 1, 0, 0), (0, 0, 0, 1) \rangle$, $H_C = \langle (0, 1, 0, 0), (0, 0, 1, 0) \rangle$, and $H_D = \langle (1, 0, 0, 0), (0, 0, 0, 1) \rangle$. In case E, if $a(v_1 + tv_2) + b(w_1 + (1/\bar{t})w_2) \in Q^4$, consideration of the first coordinate gives that a is rational. From the second coordinate we see that b is rational. Looking at the third coordinate then shows that t is necessarily rational also. Writing $t = p/q$ in lowest terms we have the rational space $\langle (1, 0, p/q, 0), (0, 1, 0, q/p) \rangle$. The associated Z summand is $H_E = \langle (q, 0, p, 0), (0, p, 0, q) \rangle$.

Summarizing, we have

LEMMA 3. *The rank 2 direct summands of $H_1(F)$ on which the Seifert form for $J\#-J^*$ vanishes are: $H_A = \langle (1, 0, 0, 0), (0, 0, 1, 0) \rangle$, $H_B = \langle (0, 1, 0, 0), (0, 0, 0, 1) \rangle$, $H_C = \langle (0, 1, 0, 0), (0, 0, 1, 0) \rangle$, $H_D = \langle (1, 0, 0, 0), (0, 0, 0, 1) \rangle$, and $H_{E,p,q} = \langle (q, 0, p, 0), (0, p, 0, q) \rangle$. (p and q relatively prime integers).*

3. Calculation of $\tau(J \# -J^*, \chi)$

In order to apply the proposition stated in Section 1 we must identify A' . It is a straightforward calculation to show that A' is generated by $\{v_1 \otimes (1/3), v_2 \otimes (1/3), w_1 \otimes (1/3), w_2 \otimes (1/3)\}$. Using Lemma 3 there are the following subgroups of $H_1(F) \otimes Q/Z$ to consider:

$$G_A = \langle v_1 \otimes (1/3), v_2 \otimes (1/3) \rangle$$

$$G_B = \langle w_1 \otimes (1/3), w_2 \otimes (1/3) \rangle$$

$$G_C = \langle v_2 \otimes (1/3), w_1 \otimes (1/3) \rangle$$

$$G_D = \langle v_1 \otimes (1/3), w_2 \otimes (1/3) \rangle$$

$$G_{E,p,q} = \langle v_1 \otimes (q/3) + v_2 \otimes (p/3), w_1 \otimes (p/3) + w_2 \otimes (q/3) \rangle$$

Notice that in the last case if either p or q is divisible by 3 (one of the two is not divisible by 3 as p and q are relatively prime) $G_{E,p,q}$ is equal to one of the other four subgroups. From now on we will assume that both p and q are relatively prime to 3.

In order to evaluate $\tau(J \# -J^*, \chi)$ we use the additivity of τ , ([4, Proposition 3.2]) $\tau(K_0 \# K_1, \chi_0 + \chi_1) = \tau(K_0, \chi_0) + \tau(K_1, \chi_1)$, and the fact that $\tau(K, 0) = 0$. The character determined by $v_1 \otimes (1/3)$ is in G_A and G_D , the character determined by $w_1 \otimes (1/3)$ is in G_B and G_C , and the character determined by $v_1 \otimes (q/3) + v_2 \otimes (p/3)$ is in $G_{E,p,q}$. Hence, one of the following conditions must hold if $J \# -J^*$ is slice.

- 1) $\tau(J, v_1 \otimes (1/3)) = 0$
- 2) $\tau(J, w_1 \otimes (1/3)) = 0$
- 3) $\tau(J, v_1 \otimes (q/3)) + \tau(-J^*, v_2 \otimes (p/3)) = 0$

According to [4, Theorem 3.5], if genus $(F) = 1$ and $\chi = x \otimes (s/m)$ with x primitive, then $\tau(K, \chi) = \rho(2\sigma_{(s/m)}(J_x) + \frac{4(m-s)s}{m^2} \theta(x, x) - \sigma_{(1/2)}(K))$, where J_x is any simple closed curve on F representing x and ρ is the injection of Q into the Witt group in which $\tau(K, \chi)$ is defined.

In the case we are studying, J is algebraically slice, so $\sigma_{1/2}(J) = \sigma_{1/2}(-J^*) = 0$. Also, $\theta(v_i, v_i) = \theta(w_i, w_i) = 0$, $i = 1$ or 2 . Finally, $J_{v_1} = L_2$, $J_{w_1} = L_1$, $J_{v_2} = -L_1$, and $J_{w_2} = -L_2$. Hence, as ρ is an injection, the 3 conditions above become:

- 1) $\sigma_{(1/3)}(L_2) = 0$
- 2) $\sigma_{(1/3)}(L_1) = 0$
- 3) $\sigma_{(q/3)}(L_2) + \sigma_{(p/3)}(-L_1) = 0$

Using the facts that $\sigma_{(1/3)}(K) = \sigma_{(2/3)}(K)$ for any knot K , and $\sigma_{(1/3)}(-K) = -\sigma_{(1/3)}(K)$, we finally achieve the result stated in Theorem 1.

The simplest example of a knot which is not concordant to its reverse

which this theorem gives is the knot which is formed when L_1 is a right handed trefoil knot and L_2 is a left handed trefoil knot.

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